

STABILITY CONDITIONS FOR PREPROJECTIVE ALGEBRAS AND ROOT SYSTEMS OF KAC-MOODY LIE ALGEBRAS

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ABSTRACT. The aim of this paper is to study the space of stability conditions on the bounded derived category of nilpotent modules over the preprojective algebra associated with a quiver without loops. We describe this space as a covering space of some open set determined by the root system of the Kac-Moody Lie algebra associated with the quiver.

1. INTRODUCTION

The notion of a stability condition on a triangulated category was introduced by T. Bridgeland in [Bri07]. He also showed that the space of all stability conditions $\text{Stab}(\mathcal{D})$ on a triangulated category \mathcal{D} has the structure of a complex manifold and there is a local isomorphism

$$\pi: \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

where $K(\mathcal{D})$ is the Grothendieck group of \mathcal{D} .

In geometric setting, the spaces of stability conditions for some 2-Calabi-Yau (CY_2) triangulated categories were studied in [Bri08, Bri09b, IUU10, Oka06]. When \mathcal{D} is the bounded derived category of coherent sheaves of a K3 surface [Bri08] or a certain subtriangulated category of coherent sheaves of a resolution of a Kleinian singularity [Bri09b], it was shown that the distinguished connected component of $\text{Stab}(\mathcal{D})$ is a covering space of some open subset of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ determined by the data of the root system and it has been clarified how the group of deck transformations acts on it. Further, the problems of connectedness and simply connectedness of $\text{Stab}(\mathcal{D})$ for some particular \mathcal{D} were solved in [IUU10, Oka06, ST01].

Other examples of CY_2 categories come from representation theories. It is known that the derived category of representations of the preprojective algebra associated with a quiver Q is CY_2 category, if Q is not of ADE type (see [Kel08]). CY_2 categories coming from Kleinian singularities also have a description in this context, as the derived categories of preprojective algebras defined by the corresponding Dynkin diagrams (see [Bri09b]).

The aim of this paper is to study the distinguished connected component of $\text{Stab}(\mathcal{D}_Q)$ for the bounded derived category \mathcal{D}_Q of the preprojective algebra associated with a quiver Q without loops (Q is not of ADE type). We describe this space as a covering space of some open set determined by the root system of the Kac-Moody Lie algebra associated with Q . This generalizes the result by Thomas for quivers of A type [Tho06] and Bridgeland for quivers of ADE type and affine ADE type [Bri09b] to arbitrary quivers without loops. Many basic ideas of this paper come from [Bri08, Bri09b].

1.1. Summary of results. Let Q be a connected finite quiver without loops (1-cycles) and let $\{1, \dots, n\}$ be the vertices of Q . Further, assume that the underlying graph \underline{Q} , which is a graph obtained by forgetting orientations of arrows in Q , is not of ADE type. For the quiver Q , we can define a \mathbb{C} -algebra $\Pi(Q)$, called the preprojective algebra of Q , which has a natural grading $\Pi(Q) = \bigoplus_{i \geq 0} \Pi(Q)_i$ by the length of paths. A right $\Pi(Q)$ -module M is called nilpotent if there is some positive integer k such that $M\Pi(Q)_l = 0$ for all $l \geq k$. Let \mathcal{A}_Q be an abelian category of finite dimensional nilpotent right A -modules and $\mathcal{D}_Q := D^b(\mathcal{A}_Q)$ be the bounded derived category of \mathcal{A}_Q . It is known that the triangulated category \mathcal{D}_Q is a CY_2 triangulated category.

The abelian category \mathcal{A}_Q is finite length with finitely many simple modules S_1, \dots, S_n corresponding to vertices of Q . Hence, every object of \mathcal{A}_Q has the Jordan-Hölder filtration by the simple modules. The Grothendieck group of \mathcal{D}_Q is given by

$$K(\mathcal{D}_Q) \cong \bigoplus_{i=1}^n \mathbb{Z}[S_i].$$

There is a natural bilinear form $\chi: K(\mathcal{D}_Q) \times K(\mathcal{D}_Q) \rightarrow \mathbb{Z}$, called the Euler form, defined by

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_Q}(E, F[i]), \quad \text{for } E, F \in \mathcal{D}_Q.$$

The CY_2 property of \mathcal{D}_Q implies that the Euler form χ is symmetric.

For Q , we associate an $n \times n$ symmetric matrix A_Q , called the generalized Cartan matrix (GCM for short), by $(A_Q)_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji})$ where q_{ij} is the number of arrows from i to j in Q . The root lattice L_Q associated with A_Q is a free abelian group with generators $\alpha_1, \dots, \alpha_n$, called simple roots:

$$L_Q := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i.$$

We also define a symmetric bilinear form $(\cdot, \cdot): L_Q \times L_Q \rightarrow \mathbb{Z}$ by $(\alpha_i, \alpha_j) := (A_Q)_{ij}$. It is known that $\chi(S_i, S_j) = (A_Q)_{ij}$, therefore we have an isomorphism between lattices

$$(K(\mathcal{D}_Q), \chi) \cong (L_Q, (\cdot, \cdot)), \quad [S_i] \mapsto \alpha_i.$$

Reflections $r_1, \dots, r_n: L_Q \rightarrow L_Q$ with respect to simple roots $\alpha_1, \dots, \alpha_n$ generates the Weyl group $W := \langle r_1, \dots, r_n \rangle$.

Let $\Delta_+^{\text{re}} \subset L_Q$ ($\Delta_+^{\text{im}} \subset L_Q$) be the set of positive real (imaginary) roots. The imaginary cone $I \subset L_Q \otimes_{\mathbb{Z}} \mathbb{R}$ is a closure of a convex hull of $\Delta_+^{\text{im}} \cup \{0\} \subset L_Q \otimes_{\mathbb{Z}} \mathbb{R}$ in the Euclidean topology as a finite dimensional vector space. Let $V := \text{Hom}_{\mathbb{Z}}(L_Q, \mathbb{C})$ and introduce the subset $X \subset V$ by

$$X := V \setminus \bigcup_{\lambda \in I \setminus \{0\}} H_{\lambda}$$

where $H_{\lambda} := \{Z \in V \mid Z(\lambda) = 0\}$. Further, define the regular subset $X_{\text{reg}} \subset V$, on which the Weyl group W acts freely, by

$$X_{\text{reg}} := X \setminus \bigcup_{\alpha \in \Delta_+^{\text{re}}} H_{\alpha}.$$

A stability condition on \mathcal{D}_Q [Bri07] consists of a pair of (Z, \mathcal{A}) ; \mathcal{A} is a full abelian subcategory $\mathcal{A} \subset \mathcal{D}_Q$, called the heart of a bounded t-structure, and Z is a group homomorphism

$$Z: K(\mathcal{D}_Q) \longrightarrow \mathbb{C},$$

called a central charge, which satisfies the condition

$$Z(E) \in \{ re^{i\pi\phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1] \}$$

for every non-zero object $E \in \mathcal{A}$. Let $\text{Stab}(\mathcal{D}_Q)$ be the set of stability conditions on \mathcal{D}_Q with the additional condition, called the support property (see Definition 4.2).

The main result in [Bri07] implies that $\text{Stab}(\mathcal{D}_Q)$ has the structure of a complex manifold and there is a local isomorphism map

$$\pi: \text{Stab}(\mathcal{D}_Q) \longrightarrow \text{Hom}_{\mathcal{D}_Q}(K(\mathcal{D}_Q), \mathbb{C}), \quad (Z, \mathcal{A}) \mapsto Z$$

on some open subset of $\text{Hom}_{\mathcal{D}_Q}(K(\mathcal{D}_Q), \mathbb{C})$. Note that under the identification $K(\mathcal{D}_Q) \cong L_Q$, we have $\text{Hom}_{\mathcal{D}_Q}(K(\mathcal{D}_Q), \mathbb{C}) \cong V$. For the space $\text{Stab}(\mathcal{D}_Q)$, there is a distinguished connected component $\text{Stab}^\circ(\mathcal{D}_Q) \subset \text{Stab}(\mathcal{D}_Q)$ which contains stability conditions with the heart \mathcal{A}_Q .

Seidel and Thomas [ST01] defined autoequivalences $\Phi_{S_1}, \dots, \Phi_{S_n} \in \text{Aut}(\mathcal{D}_Q)$, called spherical twists, for spherical objects S_1, \dots, S_n , and showed that they satisfy braid relations. Denote by $\text{Br}(\mathcal{D}_Q) \subset \text{Aut}(\mathcal{D}_Q)$ the subgroup generated by these spherical twists. The action of $\text{Br}(\mathcal{D}_Q)$ on $\text{Stab}(\mathcal{D}_Q)$ preserves the distinguished connected component $\text{Stab}^\circ(\mathcal{D}_Q)$.

The following theorem is the main result of this paper. This generalizes the results by Bridgeland and Thomas [Bri09b, Tho06] for root systems of finite or affine type to arbitrary root systems of symmetric Kac-Moody Lie algebras.

Theorem 1.1. *There is a covering map*

$$\underline{\pi}: \text{Stab}^\circ(\mathcal{D}_Q) \longrightarrow X_{\text{reg}}/W$$

and the subgroup $\mathbb{Z}[2] \times \text{Br}(\mathcal{D}_Q) \subset \text{Aut}(\mathcal{D}_Q)$ acts as the group of deck transformations ($\mathbb{Z}[2] \subset \text{Aut}(\mathcal{D}_Q)$ is the subgroup generated by the shift functor $[2] \in \text{Aut}(\mathcal{D}_Q)$).

By the van der Lek's result [vdL83], the fundamental group of X_{reg}/W is given by

$$\pi_1(X_{\text{reg}}/W) \cong \mathbb{Z}[\gamma] \times G_W$$

where $G_W = \langle \sigma_1, \dots, \sigma_n \rangle$ is the Artin group [BS72] with generators $\sigma_1, \dots, \sigma_n$ associated with the Weyl group $W = \langle r_1, \dots, r_n \rangle$. The factor $\mathbb{Z}[\gamma]$ is generated by a loop γ around the orthogonal hyperplanes of the imaginary cone $I \setminus \{0\}$. Theorem 1.1 implies that there is a surjective group homomorphism

$$\tilde{\rho}: \mathbb{Z}[\gamma] \times G_W \rightarrow \mathbb{Z}[2] \times \text{Br}(\mathcal{D}_Q).$$

We can show that $\tilde{\rho}$ sends the generators $\sigma_1, \dots, \sigma_n$ to the spherical twists $\Phi_{S_1}, \dots, \Phi_{S_n}$ and γ to the shift functor $[2]$.

The automorphism group $\text{Aut}(Q)$ of the graph Q acts on \mathcal{A}_Q by permutating simple modules $S_1, \dots, S_n \in \mathcal{A}_Q$ corresponding to vertices of Q .

Let $\text{Aut}^\circ(\mathcal{D}_Q) \subset \text{Aut}(\mathcal{D}_Q)$ be the subgroup of autoequivalences which preserve the distinguished connected component $\text{Stab}^\circ(\mathcal{D}_Q)$. Further, write by $\text{Aut}_*^\circ(\mathcal{D}_Q) =$

$\text{Aut}^\circ(\mathcal{D}_Q)/\text{Nil}^\circ(\mathcal{D}_Q)$ the quotient of $\text{Aut}^\circ(\mathcal{D}_Q)$ by the subgroup $\text{Nil}^\circ(\mathcal{D}_Q)$ consisting of autoequivalences which acts trivially on $\text{Stab}^\circ(\mathcal{D}_Q)$.

Corollary 1.2. *The group $\text{Aut}_*^\circ(\mathcal{D}_Q)$ is given by*

$$\text{Aut}_*^\circ(\mathcal{D}_Q) \cong \mathbb{Z}[1] \times (\text{Br}(\mathcal{D}_Q) \rtimes \text{Aut}(\underline{Q}))$$

where $\text{Aut}(\underline{Q})$ acts on $\text{Br}(\mathcal{D}_Q)$ by permutating the generators $\Phi_{S_1}, \dots, \Phi_{S_n}$.

When Q is of affine ADE type, similar results was shown by Corollary 1.5 in [Bri09b].

1.2. Further problems. Similar to the case of K3 surfaces in [Bri08] or Kleinian singularities in [Bri09b] (which correspond to quivers of finite or affine type), we expect the following properties for the space $\text{Stab}(\mathcal{D}_Q)$.

Conjecture 1.3. (1) *The space $\text{Stab}(\mathcal{D}_Q)$ is connected. Hence $\text{Stab}^\circ(\mathcal{D}_Q) = \text{Stab}(\mathcal{D}_Q)$.*

(2) *The space $\text{Stab}^\circ(\mathcal{D}_Q)$ is simply connected. In other words, the surjective group homomorphism*

$$\rho: G_W \longrightarrow \text{Br}(\mathcal{D}_Q)$$

is injective. (Hence ρ is an isomorphism.)

Conjecture 1.3 (1) was solved for \hat{A}_1 -quiver in [Oka06], and for A_n -quivers and \hat{A}_n -quivers in [IUU10]. Conjecture 1.3 (2) was solved for A_n -quivers in [ST01], for ADE-quivers in [BT11], and for \hat{A}_n -quivers in [IUU10].

Further, the $K(\pi, 1)$ conjecture for Artin groups (see [Par]) together with above two conjectures implies that the space $\text{Stab}(\mathcal{D}_Q)$ is contractible.

Note that if both Conjecture 1.3 (1) and (2) are true, then the autoequivalence group of \mathcal{D}_Q is given by

$$\text{Aut}(\mathcal{D}_Q) \cong \mathbb{Z}[1] \times (\text{Br}(\mathcal{D}_Q) \rtimes \text{Aut}(\underline{Q})).$$

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2. ROOT SYSTEMS OF SYMMETRIC KAC-MOODY LIE ALGEBRAS

2.1. Root lattices and Weyl groups. In this section, we recall basic notions and results for a root system associated with a generalized Cartan matrix and describe the Weyl group action on it. Further, the sets of real roots and imaginary roots are given. We refer to Kac's book and paper [Kac90, Kac78] for more details.

A matrix $A = (a_{ij})_{i,j=1}^n$ is called a generalized Cartan matrix (GCM for short) if A satisfies

- (C1) $a_{ii} = 2$ for $i = 1, \dots, n$,
- (C2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
- (C3) $a_{ij} = 0 \Rightarrow a_{ji} = 0$.

In this paper, we treat only symmetric GCMs, so the condition (C3) always true.

A matrix A is decomposable if A is a block diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

up to reordering of indices and indecomposable if otherwise.

We associate with $A = (a_{ij})_{i,j=1}^n$ a graph $S(A)$, called the Dynkin diagram of A as follows. The graph $S(A)$ has the set of vertices $\{1, \dots, n\}$, and distinct two vertices $i \neq j$ are connected by $|a_{ij}|$ edges. It is clear that A is indecomposable if and only if the graph $S(A)$ is connected.

For a real column vector $u = (u_1, u_2, \dots)^T$, the notation $u > 0$ means $u_i > 0$ for all u_i , and $u \geq 0$ means $u_i \geq 0$ for all u_i . The next result gives the classification of indecomposable GCMs.

Theorem 2.1 ([Kac90], Theorem 4.3). *Let A be an indecomposable GCM. Then one and only one of the following three possibilities holds:*

- (Fin) $\det A \neq 0$; there exists $u > 0$ such that $Au > 0$; $Au \geq 0 \Rightarrow u > 0$ or $u = 0$,
- (Aff) $\text{rank} A = n - 1$; there exists $u > 0$ such that $Au = 0$; $Au = 0 \Rightarrow u = 0$,
- (Ind) there exists $u > 0$ such that $Au < 0$; $u \geq 0 \Rightarrow u = 0$.

Referring to cases (Fin), (Aff), or (Ind), we shall say that A is of finite, affine, or indefinite type, respectively.

The root lattice associated with A is a free abelian group $L := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ with generators $\alpha_1, \dots, \alpha_n$, called simple roots. Denote by $\Pi := \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. Since A is symmetric, we can define a symmetric bilinear form $(,) : L \times L \rightarrow \mathbb{Z}$ by $(\alpha_i, \alpha_j) := a_{ij}$.

Define simple reflections $r_i : L \rightarrow L$ ($i = 1, \dots, n$) by

$$r_i(\lambda) := \lambda - (\lambda, \alpha_i)\alpha_i, \quad \text{for } \lambda \in L.$$

The group $W := \langle r_1, \dots, r_n \rangle$ generated by these reflections is called the Weyl group and satisfies the following relations (see Chapter 3 in [Kac90]):

$$r_i^2 = 1$$

and for $i \neq j$,

$$\begin{aligned} r_i r_j &= r_j r_i & \text{if } a_{ij} &= 0 \\ r_i r_j r_i &= r_j r_i r_j & \text{if } a_{ij} &= -1. \end{aligned}$$

Note that the symmetric bilinear form $(,)$ is invariant under the W -action; $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for any $\alpha, \beta \in L$ and $w \in W$.

For $\alpha = \sum_i k_i \alpha_i$, the support of α is defined to be the full subgraph $\text{supp}(\alpha) \subset S(A)$ whose vertices are $\{i \mid k_i \neq 0\} \subset \{1, \dots, n\}$.

Let $L_+ := \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$ and $L_- := -(L_+) = \sum_{i=1}^n \mathbb{Z}_{\leq 0} \alpha_i$. We define the set of (positive or negative) real roots and imaginary roots by using the W -action on L .

Definition 2.2. (1) *The set of real roots Δ^{re} is defined to be the W -orbits of simple roots Π :*

$$\Delta^{\text{re}} := W(\Pi) = \{w(\alpha_i) \mid w \in W, i = 1, \dots, n\}.$$

The set of positive real roots Δ_+^{re} (negative real roots Δ_-^{re}) is given by

$$\Delta_+^{\text{re}} := \Delta^{\text{re}} \cap L_+ \quad (\Delta_-^{\text{re}} := \Delta^{\text{re}} \cap L_-).$$

(2) Define the fundamental set of positive imaginary roots K by

$$K := \{\alpha \in L_+ \setminus \{0\} \mid \text{supp}(\alpha) \text{ is connected in } S(A), (\alpha, \alpha_i) \leq 0 \text{ for } i = 1, \dots, n\}.$$

The set of positive imaginary roots Δ_+^{im} is defined to be the W -orbits of K :

$$\Delta_+^{\text{im}} := W(K) = \{w(\alpha) \mid w \in W, \alpha \in K\}.$$

The set of negative imaginary roots is given by $\Delta_-^{\text{im}} := -\Delta_+^{\text{im}}$ and the set of all imaginary roots is given by $\Delta^{\text{im}} := \Delta_+^{\text{im}} \cup \Delta_-^{\text{im}}$.

Remark 2.3. (1) For an indecomposable GCM A , Theorem 2.1 implies that the fundamental set K is non-empty if and only if A is of affine or indefinite type. Hence the set of imaginary roots Δ^{im} is also non-empty if and only if A is of affine or indefinite type.

(2) Since K is closed under the multiplication of positive integers $\mathbb{Z}_{\geq 1}$ and the W -action commutes with it, the set Δ_+^{im} is also closed under the multiplication of $\mathbb{Z}_{\geq 1}$.

For a GCM $A = (a_{ij})_{i,j=1}^n$, take a subset $J \subset \{1, \dots, n\}$ and consider a submatrix $A_J := (a_{ij})_{i,j \in J}$. Since A_J is also a GCM, we can define the root lattice L_J , the Weyl group W_J and the set of roots Δ_J associated with A_J . By the definitions, we can naturally embed L_J and W_J into L and W by $L_J := \bigoplus_{j \in J} \mathbb{Z}\alpha_j \subset L$ and $W_J := \langle r_j \mid j \in J \rangle \subset W$. For the set of roots $\Delta_J \subset L_J$, we can easily see that $\Delta_J = L_J \cap \Delta$.

2.2. The regular subset X_{reg} . Let A be a GCM and $L = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ be the root lattice associated with A . Throughout this paper, we fix the following notations:

$$\begin{aligned} V_{\mathbb{R}}^* &:= L \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{i=1}^n \mathbb{R}\alpha_i \\ V^* &:= L \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{i=1}^n \mathbb{C}\alpha_i = V_{\mathbb{R}}^* \oplus iV_{\mathbb{R}}^* \\ V_{\mathbb{R}} &:= \text{Hom}_{\mathbb{Z}}(L, \mathbb{R}) \\ V &:= \text{Hom}_{\mathbb{Z}}(L, \mathbb{C}) = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}. \end{aligned}$$

We fix the real structures of complex vector spaces V and V^* as in the above notations.

The action of the Weyl group W on L is linearly extended to the action on $V^* \cong L \otimes_{\mathbb{Z}} \mathbb{C}$ (or $V_{\mathbb{R}}^* \otimes_{\mathbb{Z}} \mathbb{R}$) and the contragradient action of W on V (or $V_{\mathbb{R}}$) is defined by $\langle w(Z), \lambda \rangle := \langle Z, w^{-1}(\lambda) \rangle$ for $Z \in V$ and $\lambda \in V^*$.

We fix some norms for these vector spaces and consider the Euclidean topology as finite dimensional vector spaces.

Next we introduce the notion of an imaginary cone which plays central role in this paper.

Definition 2.4. Assume that the set of positive imaginary roots Δ_+^{im} is non-empty. Then, the imaginary cone I is defined to be the closure of the convex hull of the set $\Delta_+^{\text{im}} \cup \{0\}$ in $V_{\mathbb{R}}^*$. We set $I_0 := I \setminus \{0\}$ and also call I_0 the imaginary cone.

For the case Δ_+^{im} is empty, we define that I is empty.

Note that by Remark 2.3, the imaginary cone I associated to an indecomposable GCM A is non-empty if and only if A is of affine or indefinite type.

Lemma 2.5. Assume that the imaginary cone I is non-empty. Then, I is a convex cone supported on $\sum_{i=1}^n \mathbb{R}_{\geq 0}\alpha_i$.

Proof. The convexity of I is clear by the definition of I . Since $\Delta_+^{\text{im}} \cup \{0\}$ is closed under the multiplication of $\mathbb{Z}_{\geq 0}$ (see Remark 2.3), I is closed under the multiplication of $\mathbb{R}_{\geq 0}$. Further $\Delta_+^{\text{im}} \cup \{0\}$ is supported on $\sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$, therefore I is supported on $\sum_{i=1}^n \mathbb{R}_{\geq 0} \alpha_i$. \square

The following property of the imaginary cone is important in the proof of Proposition 5.5.

Proposition 2.6 ([Kac78], Proposition 1.8). *The set of rays of imaginary roots $\{\mathbb{R}_{>0}\alpha \mid \alpha \in \Delta_+^{\text{im}}\}$ is dense in the imaginary cone I_0 .*

For the compatibility of the notations in the later, we denote the canonical pairing of $Z \in V$ and $\lambda \in V^*$ by $Z(\lambda) := \langle Z, \lambda \rangle$.

Now we introduce the open subset $X \subset V$ and $X_{\text{reg}} \subset V$ as follows.

Definition 2.7. Let $\lambda \in V^*$ and $H_\lambda := \{Z \in V \mid Z(\lambda) = 0\}$ denotes a complex orthogonal hyperplane with respect to λ . A subset $X \subset V$ is defined by

$$X := V \setminus \bigcup_{\lambda \in I_0} H_\lambda$$

and a regular subset $X_{\text{reg}} \subset X \subset V$ is defined by

$$X_{\text{reg}} := X \setminus \bigcup_{\alpha \in \Delta_+^{\text{re}}} H_\alpha.$$

Since the W -action on V^* preserves real roots Δ^{re} and the imaginary cone I_0 , the W -action on X and X_{reg} is well-defined.

The following lemma is used to prove that X_{reg} is open in V .

Lemma 2.8 ([Kac90], Lemma 5.8). *In $V_{\mathbb{R}}^* \setminus \{0\}$, the limit rays for the set of rays $\{\mathbb{R}_{>0}\alpha \mid \alpha \in \Delta_+^{\text{re}}\}$ lie in I_0 .*

Define an imaginary convex disk by

$$D := I \cap \{k_1 \alpha_1 + \cdots + k_n \alpha_n \in V_{\mathbb{R}}^* \mid k_1 + \cdots + k_n = 1\}.$$

Lemma 2.5 implies that D is a compact convex subset of $V_{\mathbb{R}}^*$.

Note that since $I_0 = \mathbb{R}_{>0}D$, the subset X can be written by

$$X := V \setminus \bigcup_{\lambda \in D} H_\lambda.$$

Lemma 2.9. *The subsets $X \subset V$ and $X_{\text{reg}} \subset V$ are open in V .*

Proof. First, we say that the subset $X = V \setminus \bigcup_{\lambda \in D} H_\lambda$ is open.

Since D is compact, the projectivization $\bigcup_{\lambda \in D} \mathbb{P}H_\lambda \subset \mathbb{P}V$ is also compact. Hence, the subset $\bigcup_{\lambda \in D} H_\lambda$ is closed in V .

Let $Z \in X_{\text{reg}}$ and take a small open neighborhood $Z \in B \subset X$. Lemma 2.8 implies that the accumulated points of the set $\bigcup_{\alpha \in \Delta_+^{\text{re}}} H_\alpha$ are contained in $\bigcup_{\lambda \in D} H_\lambda$. Hence if we take B to be sufficiently small, B does not intersect the set $\bigcup_{\alpha \in \Delta_+^{\text{re}}} H_\alpha$, and this implies that X_{reg} is open. \square

Lemma 2.10. *Assume that $I_0 \subset V^*$ is non-empty. Let $Z \in X$ and consider the linear map $Z: V^* \rightarrow \mathbb{C}$. Then the image of the imaginary cone $Z(I_0) \subset \mathbb{C}$ takes the following form:*

$$Z(I_0) = \{re^{i\pi\phi} \mid r > 0 \text{ and } \phi_1 \leq \phi \leq \phi_2\}$$

where $\phi_1, \phi_2 \in \mathbb{R}$ (determined up to modulo $2\mathbb{Z}$) with $0 \leq \phi_2 - \phi_1 < 1$.

Proof. Let D be an imaginary convex disk defined above, which is compact convex. Since $Z: V^* \rightarrow C$ is \mathbb{R} -linear, the image $Z(D) \subset \mathbb{C}$ is also compact convex and $Z(I_0) = \mathbb{R}_{>0}Z(D)$. We also note that $0 \notin Z(I_0)$ since $Z \in X$. Thus the set $Z(D)$ is a compact convex subset of $\mathbb{C} \setminus \{0\}$.

First we consider the case $Z(D) \cap \mathbb{R}_{>0} = \emptyset$. Let $\text{Arg } z \in (0, 2\pi)$ for $z \in \mathbb{C} \setminus \mathbb{R}_{>0}$ the principal argument. Then, by the compactness of $Z(D)$, we have the maximum and the minimum argument of $Z(D)$ by

$$\begin{aligned}\phi_1 &:= \min \{ (1/\pi) \text{Arg } z \in (0, 2) \mid z \in Z(D) \} \\ \phi_2 &:= \max \{ (1/\pi) \text{Arg } z \in (0, 2) \mid z \in Z(D) \}.\end{aligned}$$

Clearly $0 \leq \phi_2 - \phi_1$, and $\phi_2 - \phi_1 < 1$ easily follows from the convexity of $Z(D)$ and $0 \notin Z(D)$.

In the rest case $Z(D) \cap \mathbb{R}_{>0} \neq \emptyset$, the convexity of $Z(D)$ and $0 \notin Z(D)$ implies that $Z(D) \cap \mathbb{R}_{<0} = \emptyset$. Therefore the similar argument of the first case works for the principal argument $\text{Arg } z \in (-\pi, \pi)$ for $z \in \mathbb{C} \setminus \mathbb{R}_{<0}$. \square

By using the phases ϕ_1 and ϕ_2 which appear in Lemma 2.10, we introduce the phase of the imaginary cone $\phi^I(Z)$ with respect to $Z \in X$, up to modulo $2\mathbb{Z}$, by

$$\phi^I(Z) := \frac{\phi_2 + \phi_1}{2}.$$

In the case I_0 is non-empty, define the normalized regular subset X_{reg}^N by

$$X_{\text{reg}}^N := \{ Z \in X_{\text{reg}} \mid \phi^I(Z) = 1/2 \}.$$

By the free action of $S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ on X_{reg} , we have the decomposition

$$X_{\text{reg}} \cong S^1 \times X_{\text{reg}}^N.$$

Note that the W -action on X_{reg} preserves the subset X_{reg}^N since the W -action commutes with the S^1 -action on X_{reg} .

2.3. Fundamental domain. In this section, we determine the fundamental domain for the action of the Weyl group W on the regular subset X_{reg} . To do it, we recall some basic properties of the Tits cones. Throughout this section, we assume that the GCM A is indecomposable.

Recall from Section 2.2 that V has the real structure $V_{\mathbb{R}} \subset V$ and hence $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$. Corresponding to the real structure, $Z \in V$ is written by $Z = Z_R + iZ_I$ where $Z_R, Z_I \in V_{\mathbb{R}}$.

We set $X_{\mathbb{R}} := X \cap V_{\mathbb{R}}$ and $X_{\mathbb{R}, \text{reg}} := X_{\text{reg}} \cap V_{\mathbb{R}}$. If A is of finite type, then $I = \emptyset$, therefore $X = V$ and $X_{\mathbb{R}} = V_{\mathbb{R}}$. If A is of affine or indefinite type, then $X_{\mathbb{R}}$ decomposes into two connected components

$$\begin{aligned}X_{\mathbb{R}}^+ &= \{ Z_R \in X_{\mathbb{R}} \mid Z_R(\lambda) > 0 \text{ for all } \lambda \in I_0 \} \\ X_{\mathbb{R}}^- &= \{ Z_R \in X_{\mathbb{R}} \mid Z_R(\lambda) < 0 \text{ for all } \lambda \in I_0 \}.\end{aligned}$$

These subsets are known as the Tits cones.

Definition 2.11 ([Kac90], Section 3.12). *Define the subset $C_{\mathbb{R}} \subset V_{\mathbb{R}}$, called the Weyl chamber, by $C_{\mathbb{R}} := \{ Z_R \in V_{\mathbb{R}} \mid Z_R(\alpha_i) > 0 \text{ for } i = 1, \dots, n \} (\cong \mathbb{R}_{>0}^n)$ and let $\overline{C_{\mathbb{R}}} = \{ Z_R \in V_{\mathbb{R}} \mid Z_R(\alpha_i) \geq 0 \text{ for } i = 1, \dots, n \} (\cong \mathbb{R}_{\geq 0}^n)$ be the closure of $C_{\mathbb{R}}$ in $V_{\mathbb{R}}$.*

The Tits cone $T_{\mathbb{R}}$ is defined by

$$T_{\mathbb{R}} := \bigcup_{w \in W} w(\overline{C_{\mathbb{R}}}),$$

and the regular subset of $T_{\mathbb{R}}$ is defined by

$$T_{\mathbb{R}, \text{reg}} := \bigcup_{w \in W} w(C_{\mathbb{R}}) = T_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta_+^{\text{re}}} H_{\alpha}.$$

Lemma 2.12 ([Kac90], Proposition 3.12 and Section 5.8). *The following equality holds:*

$$T_{\mathbb{R}} = \begin{cases} V_{\mathbb{R}} & \text{if } A \text{ is of finite type} \\ X_{\mathbb{R}}^+ \cup \{0\} & \text{if } A \text{ is of affine or indefinite type.} \end{cases}$$

Further, the Tits cone $T_{\mathbb{R}}$ is a convex cone.

Remark 2.13 ([Kac90], Proposition 3.12). *It is known that the closure of the Weyl chamber $\overline{C_{\mathbb{R}}}$ is a fundamental domain for the W -action on $T_{\mathbb{R}}$, and the W -action on $T_{\mathbb{R}, \text{reg}}$ is free and properly discontinuous.*

The next lemma shall be used in the proof of Proposition 2.15.

Lemma 2.14. *Assume that a GCM A is of finite type. Then for any $Z \in T_{\mathbb{R}, \text{reg}}$, there is a unique element $w \in W$ such that $w \cdot Z$ satisfies $(w \cdot Z)(\alpha_i) < 0$ for all $i = 1, \dots, n$.*

Proof. In the case that A is of finite type, Lemma 2.12 implies that

$$T_{\mathbb{R}, \text{reg}} = V_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta_+^{\text{re}}} H_{\alpha}.$$

Therefore the chamber $-C_{\mathbb{R}} = \{Z_R \in V_R \mid Z_R(\alpha_i) < 0 \text{ for } i = 1, \dots, n\}$ is contained in $T_{\mathbb{R}, \text{reg}}$, and is the fundamental domain for the action of W on $T_{\mathbb{R}, \text{reg}}$. \square

In the rest of this section, we assume that a GCM A is of affine or indefinite type. Let $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ be the upper half plane and define the semi-closed upper half plane by

$$H := \mathbb{H} \cup \mathbb{R}_{<0} = \{re^{i\pi\phi} \in \mathbb{C} \mid r > 0, \phi \in (0, 1]\}.$$

Define the normalized complexified Weyl chamber by $C^N := \{Z \in X_{\text{reg}}^N \mid Z(\alpha_i) \in H \text{ for } i = 1, \dots, n\}$.

The following is the main result of this section.

Proposition 2.15. *For any $Z \in X_{\text{reg}}^N$, there is an element $w \in W$ such that $w \cdot Z$ lies in $C^N \subset X_{\text{reg}}^N$.*

Proof. Let $Z \in X_{\text{reg}}$. Using the real structure $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$, we write $Z = Z_R + iZ_I$ where $Z_R, Z_I \in V_{\mathbb{R}}$. Since $\phi^I(Z) = 1/2$, the image $Z(I_0)$ is contained in \mathbb{H} and this implies that $Z_I(\lambda) > 0$ for all $\lambda \in I_0$. Hence we have $Z_I \in X_{\mathbb{R}}^+ = T_{\mathbb{R}} \setminus \{0\}$.

Since $\overline{C_{\mathbb{R}}}$ is the fundamental domain for the action of W on $T_{\mathbb{R}}$, there is some $w' \in W$ such that $w' \cdot Z_I \in \overline{C_{\mathbb{R}}}$.

Hence $w' \cdot Z = w' \cdot Z_R + iw' \cdot Z_I$ lies in $V_{\mathbb{R}} + i\overline{C_{\mathbb{R}}}$.

We set $Z' := w \cdot Z$, and define the subset $J \subset \{1, \dots, n\}$ by

$$J := \{j \mid Z'(\alpha_j) \in \mathbb{R}\}.$$

For J , we consider the submatrix A_J as in Section 2.1, and decompose A_J into indecomposable GCMs A_{J_1}, \dots, A_{J_l} .

Then, we can say that these indecomposable GCMs A_{J_1}, \dots, A_{J_l} are of finite type by the following reason. If A_{J_m} ($m = 1, \dots, l$) is of affine or indefinite type, the image of the imaginary cone $I_0^{J_m} \subset I_0$ is also contained in upper half plane; $Z'(I_0^{J_m}) \subset \mathbb{H}$. But by the definition of J , $Z'(\alpha_j) \in \mathbb{R}$ for all $j \in J$ and this implies that $Z'(I_0^{J_m}) \subset \mathbb{R}$. This gives the contradiction, therefore A_{J_m} is of finite type.

For $Z: L \rightarrow \mathbb{C}$, consider the restriction

$$Z'|_{L_{J_m}}: L_{J_m} \longrightarrow \mathbb{C}$$

where $L_{J_m} \subset L$ is the root lattice associated with A_{J_m} .

Since $Z'(\alpha_j) \in \mathbb{R}$ for $j \in J_m$, we can regard $Z'|_{L_{J_m}}$ as an element of the regular subset of the Tits cone $T_{\mathbb{R}, \text{reg}}^{J_m}$ associated with A_{J_m} . By Lemma 2.14, we can take an element $w_{J_m} \in W_{J_m}$ such that $(w_{J_m} \cdot Z')(\alpha_j) \in \mathbb{R}_{<0}$ for all $j \in J_m$.

Collect such elements $w_{J_1}, w_{J_2}, \dots, w_{J_l}$ and put $w := w_{J_1} w_{J_2} \cdots w_{J_l} w'$ (w does not depend on the order of $w_{J_1}, w_{J_2}, \dots, w_{J_l}$ since these elements commute). Then w is the desired element. \square

Corollary 2.16. *For any $Z \in X_{\text{reg}}$, there are elements $w \in W$ and $k \in \mathbb{C}^*$ such that $w \cdot k \cdot Z$ lies in C^N .*

Proof. Recall from Section 2.2 that $X_{\text{reg}} \cong S^1 \times X_{\text{reg}}^N$. Hence, for $Z \in X_{\text{reg}}$, we can take $k \in \mathbb{C}^*$ such that $Z' := k \cdot Z$ lies in X_{reg}^N . Then, the result follows from Proposition 2.15. \square

Proposition 2.17. *The W -action on X_{reg} is free and properly discontinuous. Further, the fundamental domain for this action is given by $S^1 \times C^N \subset S^1 \times X_{\text{reg}}^N \cong X_{\text{reg}}$.*

Proof. By Corollary 2.16, it is sufficient to consider the element $Z \in C^N$. By rotating Z , we assume that $Z(\alpha_i) \in \mathbb{H}$ for all $i = 1, \dots, n$. Note that the above condition implies that $Z_I \in C_{\mathbb{R}} \subset T_{\mathbb{R}, \text{reg}}$ where $Z = Z_R + iZ_I$ and $Z_R, Z_I \in V_{\mathbb{R}}$. Then, the result follows from the fact that the W -action on $T_{\mathbb{R}, \text{reg}}$ is free and properly discontinuous (see Remark 2.13). The second part immediately follows from Proposition 2.15. \square

2.4. Walls and chambers in X_{reg}^N . Here, we introduce the walls in X_{reg}^N , which are called the walls of second kind in [KS]. This structure shall be used in Section 4.3 to study the action of the braid group on the space of stability conditions.

Let $\overline{C^N}$ be a closure of C^N in X_{reg}^N . For $i = 1, \dots, n$, we define the walls $W_{i, \pm} \subset \overline{C^N}$ by

$$W_{i, +} := \{ Z \in X_{\text{reg}}^N \mid Z(\alpha_i) \in \mathbb{R}_{>0}, Z(\alpha_j) \in \mathbb{H} \text{ for } j \neq i \}$$

$$W_{i, -} := \{ Z \in X_{\text{reg}}^N \mid Z(\alpha_i) \in \mathbb{R}_{<0}, Z(\alpha_j) \in \mathbb{H} \text{ for } j \neq i \}.$$

Note that $W_{i, -} \subset C^N$, but $W_{i, +} \cap C^N = \emptyset$. However $r_i(W_{i, \pm}) = W_{i, \mp}$ and hence $W_{i, +} \subset r_i(C^N)$.

By using the W -action on X_{reg}^N , the set of walls is defined by

$$\{ w(W_{i, \pm}) \mid w \in W, i = 1, \dots, n \}.$$

Lemma 2.18. *For any $Z \in W_{i,\pm}$, there is a neighborhood $Z \in U \subset X_{\text{reg}}^N$ such that*

$$U \subset C^N \cup r_i(C^N).$$

Proof. Since $r_i(C^N \cup r_i(C^N)) = C^N \cup r_i(C^N)$ and $r_i(W_{i,\pm}) = W_{i,\mp}$, we only need to consider the case that $Z \in W_{i,-}$.

Let $Z \in W_{i,-}$. Take an open disk $D_i \subset \mathbb{C}$ centered at $Z(\alpha_i) \in \mathbb{R}_{<0}$ such that $0 \notin D_i$ and divide D_i into two pieces $D_{i,+} := \{z \in D_i \mid z \in H\}$ and $D_{i,-} := \{z \in D_i \mid z \in -\mathbb{H}\}$. Define a neighborhood of Z by $U := X_{\text{reg}}^N \cap (D_1 \times \cdots \times D_n)$ where D_j ($j \neq i$) is an open disk centered at $Z(\alpha_j) \in \mathbb{H}$ which is sufficiently small to satisfy $D_j \subset \mathbb{H}$. Then, it is easy to check that $U_{\pm} := X_{\text{reg}}^N \cap (D_1 \times \cdots \times D_{i,\pm} \times \cdots \times D_n)$ satisfy $U_+ \subset C^N$ and $U_- \subset r_i(C^N)$. Hence $U = U_+ \cup U_-$ is the desired open neighborhood. \square

2.5. The fundamental group of X_{reg} . In this section, we give a fundamental group of X_{reg}/W by using the result of the van der Lek [vdL83]. This is described in terms of the Artin group associated with a Coxeter system of the Weyl group W derived from A ([BS72]).

Definition 2.19 ([BS72]). *An Artin group G_W associated with the Weyl group W (derived from A) is defined to be the group generated by generators $\sigma_1, \dots, \sigma_n$ with the following relations:*

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } a_{ij} &= 0 \\ \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{if } a_{ij} &= -1. \end{aligned}$$

Recall the decomposition $X_{\text{reg}} \cong S^1 \times X_{\text{reg}}^N$ in Section 2.2. The Weyl group W acts trivially on the first factor S^1 , hence

$$X_{\text{reg}}/W \cong S^1 \times (X_{\text{reg}}^N/W).$$

Take a point $*$ in the interior of the chamber C^N . Let $[*]$ be the class of $*$ in X_{reg}/W .

Theorem 2.20 ([vdL83], see also [Par], Corollary 3.11). *Assume that a GCM A is of affine or indefinite type. Then, the fundamental group for X_{reg}/W is given by*

$$\pi_1(X_{\text{reg}}/W, [*]) \cong \mathbb{Z}[\gamma] \times G_W.$$

The generator γ of the first factor $\mathbb{Z}[\gamma]$ is given by the S^1 -orbit of $[]$. The generator σ_i of the second factor G_W is given by the path connecting to $*$ and $r_i(*)$ passing the wall $W_{i,\pm}$ in X_{reg}^N just once, which is a loop in X_{reg}^N/W .*

Proof. Since $X_{\text{reg}}/W \cong S^1 \times (X_{\text{reg}}^N/W)$ and $\pi(S^1) \cong \mathbb{Z}$, it is sufficient to prove that $\pi_1(X_{\text{reg}}^N/W) \cong G_W$. Define the regular subset of the complexified Tits cone by

$$T_{\text{reg}} := \{Z \in X_{\text{reg}} \mid \text{Im } Z \in T_{\mathbb{R}}\}.$$

The van der Lek's result in [vdL83] implies that $\pi_1(T_{\text{reg}}/W) \cong G_W$. Therefore we show that X_{reg}^N is homotopic to T_{reg} .

For $Z \in X_{\text{reg}}$, $\text{Im } Z \in T_{\mathbb{R}} \setminus \{0\} = X_{\mathbb{R}}^+$ is equivalent to $Z(I_0) \subset \mathbb{H}$ (see Section 2.3), so we have

$$T_{\text{reg}} = \{Z \in X_{\text{reg}} \mid Z(I_0) \subset \mathbb{H}\}.$$

Construct a deformation retract $h_t: T_{\text{reg}} \rightarrow T_{\text{reg}}$ by $h_t(Z) := Z \cdot e^{i\pi t(1/2 - \phi^I(Z))}$ where $0 \leq t \leq 1$. Then, it is easy to check $h_1(T_{\text{reg}}) = X_{\text{reg}}^N$ and $h_1 = \text{id}$ on $X_{\text{reg}}^N \subset T_{\text{reg}}$. Hence, it gives a homotopy equivalence $T_{\text{reg}} \sim X_{\text{reg}}^N$. \square

3. DERIVED CATEGORIES OF PREPROJECTIVE ALGEBRAS

In the following sections, abelian categories and triangulated categories are considered to be \mathbb{C} -linear categories.

3.1. Preprojective algebras of quivers. Let Q be a finite connected quiver without loops. We denote by Q_0 its set of vertices and Q_1 its set of arrows. An opposite quiver Q^{op} is obtained by reversing the orientation of arrows of Q . For an arrow $a: i \rightarrow j \in Q_1$, we denote the opposite arrow by $a^*: j \rightarrow i \in Q_1^{\text{op}}$.

A double quiver \overline{Q} is defined by adding all opposite arrows to Q , so $\overline{Q}_1 = Q_1 \cup Q_1^{\text{op}}$. For a quiver Q , define an adjacent matrix (q_{ij}) of Q by

$$q_{ij} := |\{\text{arrows from } i \text{ to } j\}|.$$

A GCM A_Q associated with Q is defined by

$$(A_Q)_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji}).$$

For a connected quiver Q , we say that Q is of finite, affine or indefinite type if the corresponding GCM A_Q is of finite, affine or indefinite type respectively (see Section 2.1).

It is known that A_Q is of finite type if and only if Q is of ADE type and A_Q is of affine type if and only if Q is of affine ADE type.

Let us denote by $\mathbb{C}Q$ a path algebra of Q over \mathbb{C} . We put a gradation on $\mathbb{C}Q$ by using the length of paths.

Definition 3.1. *The preprojective algebra $\Pi(Q)$ associated to Q is defined by*

$$\Pi(Q) := \mathbb{C}\overline{Q}/(\rho)$$

where (ρ) is an ideal of $\mathbb{C}\overline{Q}$ generated by the element

$$\rho := \sum_{a \in Q_1} (aa^* - a^*a).$$

Since ρ is a homogeneous element in $\mathbb{C}\overline{Q}$, the preprojective algebra $\Pi(Q)$ is also a graded algebra by the length of paths.

Let $\mathcal{A}_Q := \text{mod-}\Pi(Q)$ be an abelian category of finite dimensional nilpotent right $\Pi(Q)$ -modules and $K(\mathcal{A}_Q)$ be its Grothendieck group. Let S_1, \dots, S_n be simple modules corresponding to the vertices $Q_0 = \{1, \dots, n\}$. Then the Grothendieck group $K(\mathcal{A}_Q)$ is a free abelian group

$$K(\mathcal{A}_Q) \cong \oplus_{i=1}^n \mathbb{Z}[S_i]$$

generated by $[S_1], \dots, [S_n]$ which are classes of simple modules in $K(\mathcal{A}_Q)$.

3.2. Derived categories of preprojective algebras. Let Q be a finite connected quiver. In this section, we consider the bounded derived category of finite dimensional nilpotent modules of the preprojective algebra $\Pi(Q)$.

Definition 3.2. A triangulated category \mathcal{D} is called N -Calabi-Yau (CY_N) if for any objects $E, F \in \mathcal{D}$ there is a natural isomorphism

$$\nu_{E,F}: \text{Hom}_{\mathcal{D}}(E, F) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(F, E[N])^*$$

where $E[N]$ is the N -th shift of an object E and $*$ means the dual complex vector space. In other words, the N -th shift functor $[N]$ is a Serre functor.

For $E, F \in \mathcal{D}$, we write $\text{Hom}_D^i(E, F) := \text{Hom}_D(E, F[i])$.

Proposition 3.3 ([Kel08], Section 4). Let $D^b(\mathcal{A}_Q)$ be the bounded derived category of \mathcal{A}_Q . If Q is not of finite type, then $D^b(\mathcal{A}_Q)$ is a CY_2 triangulated category.

For simplicity, we write $\mathcal{D}_Q := D^b(\mathcal{A}_Q)$. Since \mathcal{D}_Q is bounded, the Grothendieck group $K(\mathcal{D}_Q)$ of the derived category \mathcal{D}_Q is isomorphic to $K(\mathcal{A}_Q)$ which is defined in the last section. From now on, we use the notation $K(\mathcal{D}_Q)$ instead of $K(\mathcal{A}_Q)$.

A bilinear form $\chi: K(\mathcal{D}_Q) \times K(\mathcal{D}_Q) \rightarrow \mathbb{Z}$, called the Euler form, is defined by

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_Q}^i(E, F).$$

By the CY_2 property, the Euler form χ is symmetric.

For the classes of simple modules $[S_i], [S_j] \in K(\mathcal{D}_Q)$, it is known that the Euler form is given by $\chi(S_i, S_j) = a_{ij}$ where a_{ij} is an entry of the GCM $A_Q = (a_{ij})$.

Hence we can identify the lattice $(K(\mathcal{D}_Q), \chi)$ with the root lattice $(L_Q, (\cdot, \cdot))$ associated with A_Q , through the map $[S_i] \mapsto \alpha_i$.

3.3. Seidel-Thomas braid groups. We define some autoequivalences of \mathcal{D}_Q which play important role in this paper, called spherical twists, introduced by P. Seidel and R. Thomas in [ST01].

An object $S \in \mathcal{D}_Q$ is called 2-spherical if

$$\text{Hom}_{\mathcal{D}_Q}^i(S, S) = \begin{cases} \mathbb{C} & \text{if } i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.4 ([ST01], Proposition 2.10). For a spherical object $S \in \mathcal{D}_Q$, there is an autoequivalence $\Phi_S \in \text{Aut}(\mathcal{D}_Q)$ such that there is an exact triangle

$$\text{Hom}_{\mathcal{D}_Q}^{\bullet}(S, E) \otimes S \longrightarrow E \longrightarrow \Phi_S(E)$$

for any object $E \in \mathcal{D}_Q$. The inverse functor $\Phi_S^{-1} \in \text{Aut}(\mathcal{D}_Q)$ is given by

$$\Phi_S^{-1}(E) \longrightarrow E \longrightarrow S \otimes \text{Hom}_{\mathcal{D}_Q}^{\bullet}(E, S)^*.$$

Since simple modules S_1, \dots, S_n of \mathcal{A}_Q are 2-spherical in \mathcal{D}_Q , they define spherical twists $\Phi_{S_i} \in \text{Aut}(\mathcal{D}_Q)$. The Seidel-Thomas braid group $\text{Br}(\mathcal{D}_Q)$ is defined to be the subgroup of $\text{Aut}(\mathcal{D}_Q)$ generated by spherical twists $\Phi_{S_1}, \dots, \Phi_{S_n}$:

$$\text{Br}(\mathcal{D}_Q) := \langle \Phi_1, \dots, \Phi_n \rangle.$$

Proposition 3.5 ([ST01], Theorem 1.2). *For the group $\mathrm{Br}(\mathcal{D}_Q)$, the following relations hold :*

$$\begin{aligned} \Phi_{S_i} \Phi_{S_j} &= \Phi_{S_j} \Phi_{S_i} & \text{if } \chi(S_i, S_j) &= 0 \\ \Phi_{S_i} \Phi_{S_j} \Phi_{S_i} &= \Phi_{S_j} \Phi_{S_i} \Phi_{S_j} & \text{if } \chi(S_i, S_j) &= -1. \end{aligned}$$

Corollary 3.6. *There is a surjective group homomorphism*

$$\rho: G_W \rightarrow \mathrm{Br}(\mathcal{D}_Q)$$

defined by $\sigma_i \mapsto \Phi_{S_i}$.

Note that at the Grothendieck group level, a spherical twist Φ_S induces a reflection $[\Phi_S]: K(\mathcal{D}_Q) \rightarrow K(\mathcal{D}_Q)$ given by

$$[\Phi_S]([E]) = [E] - \chi(S, E)[S]$$

and the inverse functor satisfies $[\Phi_S^{-1}] = [\Phi_S]$. In particular, under the identification $(K(\mathcal{D}_Q), \chi) \cong (L_Q, (\cdot, \cdot))$, the group $\mathrm{Br}(\mathcal{D}_Q)$ is reduced to the Weyl group W through the map $\Phi_{S_i} \mapsto r_i$.

Recall from Section 2.5 that the fundamental group of X_{reg}/W is isomorphic to $\mathbb{Z}[\gamma] \times G_W$. We can extend the above group homomorphism ρ to the following group homomorphism

$$\tilde{\rho}: \mathbb{Z}[\gamma] \times G_W \longrightarrow \mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q)$$

which sends $[\gamma]$ to the shift functor $[2] \in \mathrm{Aut}(\mathcal{D}_Q)$.

4. BRIDGELAND STABILITY CONDITIONS

In the following sections, we always assume that the Grothendieck group $K(\mathcal{D})$ of a triangulated category \mathcal{D} is free of finite rank ($K(\mathcal{D}) \cong \mathbb{Z}^n$ for some n).

4.1. The spaces of stability conditions. In this section, we recall the notion of stability conditions on triangulated categories introduced by T. Bridgeland in [Bri07] and collect some basic results for the space of stability conditions following [Bri07, Bri08, BS].

Definition 4.1. *Let \mathcal{D} be a triangulated category and $K(\mathcal{D})$ be its K -group. A stability condition $\sigma = (Z, \mathcal{P})$ on \mathcal{D} consists of a group homomorphism $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ called central charge and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ satisfying the following conditions:*

- (a) *if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,*
- (b) *for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,*
- (c) *if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ ($i = 1, 2$), then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,*
- (d) *for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers*

$$\phi_1 > \phi_2 > \cdots > \phi_m$$

and a collection of exact triangles

$$0 = E_0 \xrightarrow{\quad} E_1 \xrightarrow{\quad} E_2 \xrightarrow{\quad} \cdots \xrightarrow{\quad} E_{m-1} \xrightarrow{\quad} E_m = E$$

with $A_i \in \mathcal{P}(\phi_i)$ for all i .

It follows from the definition that the subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ are abelian categories (see Lemma 5.2. in [Bri07]). The nonzero objects of $\mathcal{P}(\phi)$ are called semistable of phase ϕ in σ , and simple objects in $\mathcal{P}(\phi)$ are called stable of phase ϕ in σ .

For a stability condition $\sigma = (Z, \mathcal{P})$, we introduce the set of semistable classes $\mathcal{C}^{\text{ss}}(\sigma) \subset K(\mathcal{D})$ by

$$\mathcal{C}^{\text{ss}}(\sigma) := \{ \alpha \in K(\mathcal{D}) \mid \text{there exists a semistable object } E \in \mathcal{D} \text{ in } \sigma \text{ such that } [E] = \alpha \}.$$

Similarly the set of stable classes $\mathcal{C}^s(\sigma)$ can be defined.

We always assume our stability conditions satisfy the additional assumption called the support property in [KS].

Definition 4.2. *Let $\|\cdot\|$ be some norm on $K(\mathcal{D}) \otimes \mathbb{R}$. A stability condition $\sigma = (Z, \mathcal{P})$ has a support property if there is a some constant $C > 0$ such that*

$$C \cdot |Z(\alpha)| > \|\alpha\|$$

for all $\alpha \in \mathcal{C}^{\text{ss}}(\sigma)$.

Remark 4.3. *For a ray $R = \mathbb{R}_{>0}\alpha \subset K(\mathcal{D}) \otimes \mathbb{R}$ (where $\alpha \in K(\mathcal{D}) \setminus \{0\}$), define a function f by*

$$f(R) := \frac{|Z(\alpha)|}{\|\alpha\|}.$$

(This doesn't depend on the choice of α , only depends on the ray.)

Let $\mathbb{R}_{>0}\mathcal{C}^{\text{ss}}(\sigma) := \{ \mathbb{R}_{>0}\alpha \mid \alpha \in \mathcal{C}^{\text{ss}}(\sigma) \}$ be the set of rays generated by semistable classes of σ . Then, the support property of $\sigma = (Z, \mathcal{P})$ is equivalent to that there is no sequence of rays $R_i \subset \mathbb{R}_{>0}\mathcal{C}^{\text{ss}}(\sigma)$ ($i = 1, 2, \dots$) such that

$$\lim_{i \rightarrow \infty} f(R_i) = 0.$$

Let $\text{Stab}(\mathcal{D})$ be the set of all stability conditions on \mathcal{D} with the support property. In [Bri07], Bridgeland introduced a natural topology on $\text{Stab}(\mathcal{D})$ induced by the metric $d: \text{Stab}(\mathcal{D}) \times \text{Stab}(\mathcal{D}) \rightarrow [0, \infty]$. For more details, we refer to Section 8 in [Bri07].

In this topology, Bridgeland showed the following crucial theorem.

Theorem 4.4 ([Bri07], Theorem 1.2). *The space $\text{Stab}(\mathcal{D})$ has the structure of a complex manifold and the projection map of central charges*

$$\pi: \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

defined by $(Z, \mathcal{P}) \mapsto Z$ is a local isomorphism onto an open subset of $\text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$.

The next lemma implies local injectivity of the above projection map π .

Lemma 4.5 ([Bri07], Lemma 6.4). *Let $\sigma = (Z, \mathcal{P})$ and $\sigma' = (Z, \mathcal{P}')$ be stability conditions on \mathcal{D} with the same central charge Z . Then, $d(\sigma, \sigma') < 1$ implies $\sigma = \sigma'$.*

The following lemma shall be used in the proof of Proposition 5.3 and Proposition 5.5.

Lemma 4.6. *Fix a class $\alpha \in K(\mathcal{D})$ and let $U \subset \text{Stab}(\mathcal{D})$ be an open subset. If every stability condition $\sigma \in U$ satisfies $\alpha \in \mathcal{C}^{\text{ss}}(\sigma)$, then a stability condition on the boundary $\sigma' \in \partial U$ also satisfies $\alpha \in \mathcal{C}^{\text{ss}}(\sigma')$.*

Proof. This follows from the results for walls and chambers in [Bri08, Section 9] or [BS, Section 7.6]. \square

4.2. Stability conditions on finite length abelian categories. In [Bri07], Bridgeland gave the alternative description of a stability condition on \mathcal{D} as the pair of a bounded t-structure and a central charge on its heart. In this section, by using this description, we construct stability conditions on finite length abelian categories with finitely many simple objects.

Definition 4.7. *Let \mathcal{A} be an abelian category and let $K(\mathcal{A})$ be its Grothendieck group. A central charge on \mathcal{A} is a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ such that for any nonzero object $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ lies in semi-closed upper half-plane $H = \{re^{i\pi\phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1]\}$.*

The real number $\phi(E) := (1/\pi) \arg Z(E) \in (0, 1]$ for $0 \neq E \in \mathcal{A}$ is called the phase of E .

A nonzero object $0 \neq E \in \mathcal{A}$ is said to be Z -(semi)stable if for any nonzero proper subobject $0 \neq A \subsetneq E$ satisfies $\phi(A) < (\leq) \phi(E)$.

Proposition 4.8 ([Bri07], Proposition 5.3). *Let \mathcal{D} be a triangulated category. To give a stability condition on \mathcal{D} is equivalent to giving the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded structure on \mathcal{D} and a central charge with the Harder-Narasimhan property on \mathcal{A} .*

For the heart of a bounded t-structure and the Harder-Narasimhan property (HN property), we refer to Section 2 and 3 in [Bri07].

We denote by $\text{Stab}(\mathcal{A})$ the set of central charges on the heart $\mathcal{A} \subset \mathcal{D}$ with the HN property and the support property.

Proposition 4.8 implies that there is a natural inclusion

$$\text{Stab}(\mathcal{A}) \subset \text{Stab}(\mathcal{D}).$$

Let $\mathcal{A} \subset \mathcal{D}$ be a finite heart, which is an abelian category with finitely many simple objects S_1, \dots, S_n and generated by means of extensions of these simple objects. Then, we have $K(\mathcal{A}) \cong \oplus_{i=1}^n \mathbb{Z}[S_i]$. For any point $(z_1, \dots, z_n) \in H^n$, the central charge $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ is defined by $Z(S_i) := z_i$. Conversely, for a given central charge $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$, the complex number $Z(S_i)$ lies in H for all i . Hence Z determines a point $(z_1, \dots, z_n) \in H^n$ where $z_i := Z(S_i)$. As a result, the set of central charges on \mathcal{A} is isomorphic to H^n .

Lemma 4.9 ([Bri09a], Lemma 5.2). *Let $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ be a central charge given by the above construction. Then Z has the HN property. In particular, we have*

$$\text{Stab}(\mathcal{A}) \cong H^n.$$

As in Section 3, let \mathcal{A}_Q be an abelian category of finite dimensional nilpotent $\Pi(Q)$ -modules and \mathcal{D}_Q be the bounded derived category of \mathcal{A}_Q . Then, there is a unique distinguished connected component $\text{Stab}^\circ(\mathcal{D}_Q) \subset \text{Stab}(\mathcal{D}_Q)$ which contains the subset $\text{Stab}(\mathcal{A}_Q)$.

4.3. Group actions on $\text{Stab}^\circ(\mathcal{D}_Q)$. Here we consider two group actions on $\text{Stab}^\circ(\mathcal{D}_Q)$, which are lift of the action of \mathbb{C}^* and W on X_{reg} . Further, we prove the lifted version of Proposition 2.15 and Corollary 2.16.

For the space $\text{Stab}(\mathcal{D})$, we introduce two group actions which commute each other.

First, consider the action of $\text{Aut}(\mathcal{D})$ on $\text{Stab}(\mathcal{D})$. Let $\Phi \in \text{Aut}(\mathcal{D})$ be an autoequivalence of \mathcal{D} and $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{D})$ be a stability condition on \mathcal{D} . Then, the

element $\Phi \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}')$ is defined by

$$Z'(E) := Z(\Phi^{-1}(E)), \quad \mathcal{P}'(\phi) := \Phi(\mathcal{P}(\phi)),$$

where $E \in \mathcal{D}$ and $\phi \in \mathbb{R}$.

Secondly, define the \mathbb{C} -action on $\text{Stab}(\mathcal{D})$. For $t \in \mathbb{C}$ and $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{D})$, the element $t \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}')$ is defined by

$$Z'(E) := e^{-i\pi t} \cdot Z(E), \quad \mathcal{P}'(\phi) := \mathcal{P}(\phi + \text{Re}(t))$$

where $E \in \mathcal{D}$ and $\phi \in \mathbb{R}$. Clearly, this action is free.

Note that by the definition, these two actions are isometries with respect to the distance d on the space $\text{Stab}(\mathcal{D})$.

On $K(\mathcal{D})$, we can easily see the following remark about semistable classes.

Remark 4.10. *By the action of an autoequivalence $\Phi \in \text{Aut}(\mathcal{D})$, the set of semistable classes $\mathcal{C}^{\text{ss}}(\sigma)$ for $\sigma \in \text{Stab}(\mathcal{D})$ changes to*

$$\mathcal{C}^{\text{ss}}(\Phi \cdot \sigma) = [\Phi](\mathcal{C}^{\text{ss}}(\sigma)).$$

On the other hand, the \mathbb{C} -action on $\text{Stab}(\mathcal{D})$ does not change the set of semistable classes $\mathcal{C}^{\text{ss}}(\sigma)$.

In the rest of this section, we study the action of the Seidel-Thomas braid group $\text{Br}(\mathcal{D}_Q)$ on $\text{Stab}^\circ(\mathcal{D}_Q)$. Recall the identification $K(\mathcal{D}_Q) \cong L_Q$ and consider the projection map

$$\pi: \text{Stab}^\circ(\mathcal{D}_Q) \longrightarrow V$$

where $V = \text{Hom}_{\mathbb{Z}}(L_Q, \mathbb{C})$ (see Section 2.2).

Then, π maps the subset $H^n \cong \text{Stab}(\mathcal{A}_Q) \subset \text{Stab}^\circ(\mathcal{D}_Q)$ isomorphically onto the subset

$$H^n \cong \{Z \in X_{\text{reg}} \mid Z(\alpha_i) \in H \text{ for } i = 1, \dots, n\} \subset V.$$

Corresponding to the normalized regular subset $X_{\text{reg}}^N \subset V$, we introduce the space of normalized stability conditions $\text{Stab}(\mathcal{D}_Q)^N$ by

$$\text{Stab}(\mathcal{D}_Q)^N := \{\sigma \in \text{Stab}^\circ(\mathcal{D}_Q) \mid \pi(\sigma) \in X_{\text{reg}}^N\},$$

and normalized stability conditions on \mathcal{A}_Q by

$$\text{Stab}(\mathcal{A}_Q)^N := \{\sigma \in \text{Stab}(\mathcal{A}_Q) \mid \pi(\sigma) \in X_{\text{reg}}^N\}.$$

Note that the projection $\pi: \text{Stab}(\mathcal{D}_Q)^N \rightarrow X_{\text{reg}}^N$ maps $\text{Stab}(\mathcal{A}_Q)^N$ isomorphically onto the chamber $C^N \subset X_{\text{reg}}^N$.

Recall from Section 2.4 that there are walls $W_{i,\pm} \subset \overline{C^N}$ for $i = 1, \dots, n$. Define the lifted walls $\widetilde{W}_{i,\pm} \subset \overline{\text{Stab}(\mathcal{A}_Q)^N}$ for $i = 1, \dots, n$ by

$$\widetilde{W}_{i,+} := \{\sigma = (Z, \mathcal{P}) \in \overline{\text{Stab}(\mathcal{A}_Q)^N} \mid Z(S_i) \in \mathbb{R}_{>0}, Z(S_j) \in \mathbb{H} \text{ for } j \neq i\}$$

$$\widetilde{W}_{i,-} := \{\sigma = (Z, \mathcal{P}) \in \overline{\text{Stab}(\mathcal{A}_Q)^N} \mid Z(S_i) \in \mathbb{R}_{<0}, Z(S_j) \in \mathbb{H} \text{ for } j \neq i\}.$$

Note that as in Section 2.4, $\widetilde{W}_{i,-} \subset \text{Stab}(\mathcal{A}_Q)^N$ but $\widetilde{W}_{i,+} \cap \text{Stab}(\mathcal{A}_Q)^N = \emptyset$. However, $\Phi_{S_i}^{-1}(\widetilde{W}_{i,-}) = \widetilde{W}_{i,+}$ and $\widetilde{W}_{i,+} \subset \Phi_{S_i}^{-1}(\text{Stab}(\mathcal{A}_Q)^N)$.

The following is the lift of Lemma 2.18.

Lemma 4.11. *Let $\sigma \in \widetilde{W}_{i,\pm} \subset \overline{\text{Stab}(\mathcal{A}_Q)^N}$. Then, there is a neighborhood $\sigma \in U \subset \text{Stab}(\mathcal{D}_Q)^N$ such that one of the following holds*

- (1) $U \subset \text{Stab}(\mathcal{A}_Q)^N \cup \Phi_{S_i}^{-1}(\text{Stab}(\mathcal{A}_Q)^N)$ if $\sigma \in \widetilde{W}_{i,+}$,

(2) $U \subset \text{Stab}(\mathcal{A}_Q)^N \cup \Phi_{S_i}(\text{Stab}(\mathcal{A}_Q)^N)$ if $\sigma \in \widetilde{W_{i,-}}$.

Proof. Note that in a CY_2 category, simple tilted categories $\mu_{S_i}^\pm(\mathcal{A}_Q)$ correspond to $\Phi_{S_i}^{\pm 1}(\mathcal{A}_Q)$. Then it follows from Lemma 5.5 in [Bri09a] or Lemma 7.9 in [BS]. \square

Lemma 4.12. *The image of the projection map $\pi: \text{Stab}^\circ(\mathcal{D}_Q) \rightarrow V$ contains X_{reg} .*

Proof. Recall from Corollary 2.16 that the orbit of $C^N \subset V$ under the action of \mathbb{C}^* and W coincides with X_{reg} . Since the action of \mathbb{C} and $\text{Br}(\mathcal{D}_Q)$ on $\text{Stab}^\circ(\mathcal{D}_Q)$ is reduced to the action of \mathbb{C}^* and W on the base space V , the orbit of $\text{Stab}(\mathcal{A}_Q)^N \subset \text{Stab}^\circ(\mathcal{D}_Q)$ under the action of \mathbb{C} and $\text{Br}(\mathcal{D}_Q)$ is mapped to the subset $X_{\text{reg}} \subset V$. \square

Let $\text{Stab}^\circ(\mathcal{D}_Q)^N$ be the connected component of $\text{Stab}(\mathcal{D}_Q)^N$ which contains $\text{Stab}(\mathcal{A}_Q)^N$. Now, we lift Proposition 2.15 and Corollary 2.16 via the restricted projection map $\pi: \pi^{-1}(X_{\text{reg}}) \rightarrow X_{\text{reg}}$.

In the proof of the following result, we use the same argument in the proof of Proposition 13.2 in [Bri08].

Proposition 4.13. *For any $\sigma \in \text{Stab}^\circ(\mathcal{D}_Q)^N$, there is an autoequivalence $\Phi \in \text{Br}(\mathcal{D}_Q)$ such that $\Phi \cdot \sigma$ lies in $\text{Stab}(\mathcal{A}_Q)^N$.*

Proof. Let $\sigma = (Z, \mathcal{P}) \in \text{Stab}^\circ(\mathcal{D}_Q)^N$ and take a path $\gamma: [0, 1] \rightarrow \text{Stab}^\circ(\mathcal{D}_Q)^N$ such that $\gamma(0) \in \text{Stab}(\mathcal{A}_Q)^N$ and $\gamma(1) = \sigma$. By Lemma 2.18, we can deform γ to satisfy that for any $t \in (0, 1)$, the path $\pi(\gamma) \subset X_{\text{reg}}^N$ passes the walls $\{w(W_{i,\pm}) \mid w \in W, i = 1, \dots, n\}$ only at $t_1, \dots, t_m \in (0, 1)$ with $0 < t_1 < \dots < t_m < 1$.

Since $\gamma([0, t_1]) \subset \text{Stab}(\mathcal{A}_Q)^N$ and $\pi(\gamma(t_1)) \in W_{i,\pm}$ for some i , the stability $\gamma(t_1) \in \text{Stab}^\circ(\mathcal{D}_Q)^N$ lies in $\widetilde{W_{i,\pm}}$. If $\gamma(t_1) \in \widetilde{W_{i,+}}$, define $\Phi_1 := \Phi_{S_{i_1}}$, and if $\gamma(t_1) \in \widetilde{W_{i,-}}$, define $\Phi_1 := \Phi_{S_{i_1}}^{-1}$. Then, by Lemma 4.11, $(\Phi_1\gamma)(t_1, t_2) \subset \text{Stab}(\mathcal{A}_Q)^N$ and $\pi((\Phi_1\gamma)(t_2)) \in W_{i_2,\pm}$. Hence $(\Phi_1\gamma)(t_2) \in \widetilde{W_{i_2,\pm}}$, and we can similarly define $\Phi_2 := \Phi_{S_{i_2}}\Phi_1$ or $\Phi_2 := \Phi_{S_{i_2}}^{-1}\Phi_1$.

Repeating this process, we get an autoequivalence $\Phi_m \in \text{Br}(\mathcal{D}_Q)$ such that $(\Phi_m\gamma)(t_m, 1) \subset \text{Stab}(\mathcal{A}_Q)^N$. \square

Let $\pi^{-1}(X_{\text{reg}})^\circ$ be the connected component of $\pi^{-1}(X_{\text{reg}})$ which contains $\text{Stab}(\mathcal{A}_Q)$.

Corollary 4.14. *For $\sigma \in \pi^{-1}(X_{\text{reg}})^\circ \subset \text{Stab}^\circ(\mathcal{D}_Q)$, there are elements $\Phi \in \text{Br}(\mathcal{D}_Q)$ and $k \in \mathbb{C}$ such that $\Phi \cdot k \cdot \sigma \in \text{Stab}(\mathcal{A}_Q)^N$.*

Proof. Let $\sigma \in \pi^{-1}(X_{\text{reg}})^\circ$ and take a path $\gamma: [0, 1] \rightarrow \pi^{-1}(X_{\text{reg}})^\circ$ such that $\gamma(0) \in \text{Stab}(\mathcal{A}_Q)$ and $\gamma(1) = \sigma$. By the \mathbb{C} -action on $\pi^{-1}(X_{\text{reg}})^\circ$, we can normalize γ to the path $\gamma' = k \cdot \gamma$ which lies in $\text{Stab}^\circ(\mathcal{D}_Q)^N$ where $k: [0, 1] \rightarrow \mathbb{C}$ and $\gamma'(t) = k(t) \cdot \gamma(t)$. Then, the result follows from Proposition 4.13. \square

5. PROOF OF MAIN THEOREM

5.1. Indivisible roots and semistable classes. In this section, we show that the set of indivisible roots are contained in the set of semistable classes of the stability condition with the central charge in X_{reg} .

Set $K(\mathcal{D}_Q)_{\geq 0} := \sum_{i=1}^n \mathbb{Z}_{\geq 0}[S_i]$ and $K(\mathcal{D}_Q)_{> 0} := K(\mathcal{D}_Q)_{\geq 0} \setminus \{0\}$. For classes $\alpha, \beta \in K(\mathcal{D}_Q)$, we write by $\alpha > \beta$ if $\alpha - \beta \in K(\mathcal{D}_Q)_{> 0}$.

In the following we fix the class $\alpha \in K(\mathcal{D}_Q)_{>0}$. We denote by $\text{Rep}(\overline{Q}, \alpha)$ the affine space consisting of representations of the double quiver \overline{Q} with the class α . Let $\text{Rep}(\Pi(Q), \alpha)^{\text{nil}}$ be the set of nilpotent $\Pi(Q)$ -modules (representations) with the class α . Lusztig showed that $\text{Rep}(\Pi(Q), \alpha)^{\text{nil}}$ is a Lagrangian subvariety of the affine space $\text{Rep}(\overline{Q}, \alpha)$ (see Section 12 in [Lus91]).

Let $Z: K(\mathcal{D}_Q) \rightarrow \mathbb{C}$ be a central charge on \mathcal{A}_Q . The central charge Z is identified with the King's stability condition $\lambda: K(\mathcal{D}_Q) \rightarrow \mathbb{R}$ [Kin94] by defining

$$\lambda(\beta) := -\text{Im} \frac{Z(\beta)}{Z(\alpha)}, \quad \beta \in K(\mathcal{D}_Q).$$

Therefore, here we use the notion of a central charge instead of the notion of a King's stability condition which is used in [CBvdB].

Following [CBvdB], we introduce generic stability conditions.

Definition 5.1. *A central charge $Z: K(\mathcal{D}_Q) \rightarrow \mathbb{C}$ is said to be generic with respect to α if $\text{Im}(Z(\beta)/Z(\alpha)) \neq 0$ for all $0 < \beta < \alpha$.*

Let $\mathfrak{g}(A_Q)$ be the Kac-Moody Lie algebra associated with the GCM A_Q (see Chapter 1 in [Kac90]). The root multiplicity of a root $\alpha \in \Delta$ is defined to be the dimension $\dim \mathfrak{g}_\alpha$ where \mathfrak{g}_α is the root space given by the decomposition $\mathfrak{g}(A_Q) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

A root $\alpha \in \Delta$ is called indivisible if there is no $\beta \in \Delta$ satisfying $\alpha = m\beta$ for $|m| > 1$.

Proposition 5.2 ([CBvdB], Proposition 1.2). *Let $\alpha \in \Delta_+$ be a positive indivisible root and suppose that Z is generic with respect to α . Then, the number of irreducible components of $\text{Rep}(\Pi(Q), \alpha)^{\text{nil}}$ which contain a Z -stable representation is equal to the root multiplicity $\dim \mathfrak{g}_\alpha$.*

As a result, it turns out that if Z is generic with respect to a positive indivisible root $\alpha \in \Delta_+$, then the moduli space of Z -stable nilpotent modules with the class α is non-empty.

The following is a key result in the proof of Proposition 5.5.

Proposition 5.3. *Let $\sigma = (Z, \mathcal{P}) \in \pi^{-1}(X_{\text{reg}})^\circ$. Then, the set of indivisible roots in Δ is contained in the set of σ -semistable classes $\mathcal{C}^{\text{ss}}(\sigma)$:*

$$\{\alpha \in \Delta \mid \alpha \text{ is indivisible}\} \subset \mathcal{C}^{\text{ss}}(\sigma).$$

Proof. First note that the set of all indivisible roots are invariant under the action of W . Since $\sigma \in \pi^{-1}(X_{\text{reg}})^\circ$, by Remark 4.10 and Corollary 4.14, it is sufficient to prove that any stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{A}_Q)$ contains all indivisible roots as semistable classes.

Fix a positive indivisible root $\alpha \in \Delta_+$ and consider the dense open subset of $\text{Stab}(\mathcal{A}_Q)$ consisting of stability conditions which are generic with respect to α :

$$\{Z \in \text{Stab}(\mathcal{A}_Q) \mid \text{Im}(Z(\beta)/Z(\alpha)) \neq 0 \text{ for all } 0 < \beta < \alpha\}.$$

By Proposition 5.2, any stability condition in this subset contains α as a stable class. Since this subset is dense in $\text{Stab}(\mathcal{A}_Q)$, by Lemma 4.6 we conclude that any stability condition in $\text{Stab}(\mathcal{A}_Q)$ contains α at least as a semistable class. \square

5.2. Projection of central charges. In this section, we determine the image of central charges via the projection map

$$\pi: \text{Stab}^\circ(\mathcal{D}_Q) \rightarrow V$$

by using Proposition 5.3 proved in the last section.

Lemma 5.4. *Let ∂X_{reg} be a boundary of X_{reg} and assume that $Z \in \partial X_{\text{reg}}$. Then, there is at least one ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\text{re}}$ such that $Z(R) = 0$.*

Proof. It immediately follows from the definition of the open subset $X_{\text{reg}} \subset V$ (see Definition 2.7). \square

Proposition 5.5. *The projection map*

$$\pi: \text{Stab}^\circ(\mathcal{D}_Q) \rightarrow V$$

maps $\text{Stab}^\circ(\mathcal{D}_Q)$ onto the subset $X_{\text{reg}} \subset V$.

Proof. One inclusion $X_{\text{reg}} \subset \pi(\text{Stab}^\circ(\mathcal{D}_Q))$ is Lemma 4.12. Here we prove the other inclusion $\pi(\text{Stab}^\circ(\mathcal{D}_Q)) \subset X_{\text{reg}}$, which is equivalent to that $\pi^{-1}(X_{\text{reg}})^\circ = \text{Stab}^\circ(\mathcal{D}_Q)$.

Since $\text{Stab}^\circ(\mathcal{D}_Q)$ is the connected component which contains $\pi^{-1}(X_{\text{reg}})^\circ$, it is sufficient to prove that $\pi^{-1}(X_{\text{reg}})^\circ$ is open and closed.

First note that since X_{reg} is open, the connected component $\pi^{-1}(X_{\text{reg}})^\circ$ is also open. Hence, the closedness of $\pi^{-1}(X_{\text{reg}})^\circ$ is equivalent to that it has no boundary points.

Assume that $\pi^{-1}(X_{\text{reg}})^\circ$ has a boundary point $\sigma = (Z, \mathcal{P})$. Then, $\sigma = (Z, \mathcal{P})$ is projected on ∂X_{reg} . Hence, by Lemma 5.4, there is a ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\text{re}}$ such that $Z(R) = 0$.

For the ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\text{re}}$, by Proposition 2.6, we can take a sequence of rays R_i ($i = 1, 2, \dots$) such that $R_i \rightarrow R$ (as $i \rightarrow \infty$) where $R_i = \mathbb{R}_{>0}\alpha_i$ and each $\alpha_i \in \Delta_+$ is an indivisible positive root.

On the other hand, since σ lies in the closure of $\pi^{-1}(X_{\text{reg}})^\circ$, by Lemma 4.6 and Proposition 5.3, σ contains all indivisible roots as semistable classes. In particular, the above rays R_i ($i = 1, 2, \dots$) are contained in $\mathbb{R}_{>0}\mathcal{C}^{\text{ss}}(\sigma)$.

Since $Z(R) = 0$, we have

$$\lim_{i \rightarrow \infty} f(R_i) = f(R) = 0$$

where f is the function defined in Remark 4.2. But this contradicts to the support property of σ (see Remark 4.2). \square

5.3. Covering structures.

Proposition 5.6. *The action of $\mathbb{Z}[2] \times \text{Br}(\mathcal{D}_Q) \subset \text{Aut}(\mathcal{D}_Q)$ on $\text{Stab}^\circ(\mathcal{D}_Q)$ is free and properly discontinuous.*

Proof. The result is clear for the action of $\mathbb{Z}[2]$. Hence we prove them for the action of $\text{Br}(\mathcal{D}_Q)$.

We first prove that the action of $\text{Br}(\mathcal{D}_Q)$ is free. By Corollary 4.14, it is sufficient to prove that for $\sigma \in \text{Stab}(\mathcal{A}_Q)^N$. Let $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{A}_Q)^N$ and suppose $(Z, \mathcal{P}) = ([\Phi]^{-1} \cdot Z, \Phi(\mathcal{P}))$. Since any object in \mathcal{D}_Q is generated by finite extensions of objects given by the shift of S_1, \dots, S_n , the isomorphism $\Phi(S_i) \cong S_i$ for all $i = 1, \dots, n$ implies $\Phi \cong \text{id}$. Assume that $S_i \in \mathcal{P}(\phi_i) \subset \mathcal{A}_Q$, then also $\Phi(S_i) \in$

$\mathcal{P}(\phi_i) \subset \mathcal{A}_Q$ since $\Phi(\mathcal{P}(\phi_i)) = \mathcal{P}(\phi_i)$. At the K -group level, $[\Phi]^{-1} \cdot Z = Z$ implies that $[\Phi] = \text{id}$, therefore we have $[\Phi(S_i)] = [S_i]$. Both $\Phi(S_i)$ and S_i are objects in \mathcal{A}_Q with the same K -group class $[S_i]$, and such an object is unique up to isomorphism in \mathcal{A}_Q . Hence $\Phi(S_i) \cong S_i$.

Next we prove that the action of $\text{Br}(\mathcal{D}_Q)$ is properly discontinuous. It is sufficient to prove that for any $\sigma \in \text{Stab}(\mathcal{A}_Q)^N$, there is some open subset U such that $U \cap \Phi \cdot U = \emptyset$ for any $\text{id} \neq \Phi \in \text{Br}(\mathcal{D}_Q)$. There are two cases either $[\Phi] = \text{id}$ or not. In the case $[\Phi] \neq \text{id}$, this immediately follows from Proposition 2.17 and the local isomorphism property of Theorem 4.4. In the case $[\Phi] = \text{id}$, the stability conditions σ and $\Phi \cdot \sigma$ are different but have the same central charge. Hence, the result follows from Lemma 4.5. \square

Denote by $\underline{\pi}$ the composition of $\pi: \text{Stab}^\circ(\mathcal{D}_Q) \rightarrow X_{\text{reg}}$ and $X_{\text{reg}} \rightarrow X_{\text{reg}}/W$. Now we prove Theorem 1.1.

Proof of Theorem 1.1. The remaining part is to show that the quotient of $\text{Stab}^\circ(\mathcal{D}_Q)$ by $\mathbb{Z}[2] \times \text{Br}(\mathcal{D}_Q)$ coincides with X_{reg}/W . This is equivalent to that for any $\sigma_1, \sigma_2 \in \text{Stab}^\circ(\mathcal{D}_Q)$, if $\pi(\sigma_1) = \pi(\sigma_2)$, then there are elements $[2n] \in \mathbb{Z}[2]$ and $\Phi \in \text{Br}(\mathcal{D}_Q)$ with $[\Phi] = \text{id}$ such that $\sigma_1 = \Phi \cdot [2n] \cdot \sigma_2$.

By Corollary 4.14, we can assume that $\sigma_1 \in \text{Stab}(\mathcal{A}_Q)^N$. Further, there are elements $k \in \mathbb{C}$ and $\Phi \in \text{Br}(\mathcal{D}_Q)$ such that $\sigma'_2 := \Phi \cdot k \cdot \sigma_2$ lies in $\text{Stab}(\mathcal{A}_Q)^N$. By mapping σ'_2 onto X_{reg} , we have

$$\pi(\sigma'_2) = [\Phi] \cdot e^{-i\pi k} \cdot \pi(\sigma_2) = [\Phi] \cdot e^{-i\pi k} \cdot \pi(\sigma_1).$$

Since $\text{Stab}(\mathcal{A}_Q)^N$ is mapped isomorphically onto the normalized chamber C^N and both $\pi(\sigma_1)$ and $\pi(\sigma'_2)$ lie in C^N , we have $[\Phi] = \text{id}$, $k = 2n \in 2\mathbb{Z}$ and $\pi(\sigma_1) = \pi(\sigma'_2)$. \square

Corollary 1.2 is proved in the completely same way as the proof of Corollary 1.5 in [Bri09b].

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